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Asymptotic symmetries and asymptotically symmetric solutions of partial differential equations

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Abstract. Symmetry methods for differential equations are a powerful tool to attack nonlinear problems, in particular for determining solutions with given symmetries to nonlinear PDEs. Since in real applications one is often interested in solutions which are *asymptotically* symmetric, we propose here an approach to *asymptotic symmetry* based on the methods of Lie theory. We adopt, translate in geometric language and develop the renormalization group approach recently proposed by Bricmont and Kupiainen for the Ginzburg–Landau equation.

1. Introduction

Application of symmetry methods in the study of differential equations—both ODEs and PDEs—has proved in recent years to be one of the most powerful ways to attack non-linear problems.

We refer to [1–3] for the general theory and applications (see also [4–6] for earlier references, and [7–10] for short introductions), as well as to [10–18] for the subject of *conditional* (or *partial*) symmetries, and will assume in the following that the reader has some familiarity with the subject.

These methods are, in particular, quite effective in determining solutions with given symmetries to non-linear PDEs. The point we would like to make in the present note, is that in real applications one is often interested not so much in solutions with a given *exact symmetry*, but rather in solutions which are *asymptotically* (in time and/or space) symmetric.

As a simple example for the occurrence of such a situation, consider the ODE $dy/dt = -2ty^2$ which is invariant under the scaling $t \to \lambda t$, $y \to \lambda^{-2}y$ and has as solution $y(t) = 1/(1+t^2)$. This solution is not invariant under the above scaling, but in the limit $t \to \pm\infty$ the scaling symmetry is asymptotically recovered.

We want indeed to propose an approach to *asymptotic symmetry* based on the above mentioned, well known methods of Lie theory (LT) approach to differential equations (DEs in the following).

Although our approach could seem original, what it is done here is merely to adopt—and translate in the LT language—the renormalization group (RG) approach recently developed by Bricmont, Kupiainen and Lin (in the following, BKL), who also studied a specific problem, namely the Ginzburg–Landau equation [19–21]. More precisely, we develop an infinitesimal RG approach in the language and framework of LT. Apart from the interest of this in itself,

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it is hoped that such an approach could later permit making a contact with recent results concerning the existence results for solutions of non-linear Yang–Mills-type equations on the basis of their symmetry properties; these results could in turn be themselves applicable to pattern-formation problems [20].

To the best of my knowledge, no mention is made of asymptotic symmetries in the literature concerned with Lie-theoretic study of differential equations; the emphasis has in general been placed on exact or on partial symmetries (see the references above). Recently, several authors have considered *approximate symmetries* [23], but no consideration was given to their asymptotic properties; moreover, as the simple example above shows, one can have asymptotic symmetries which are not symmetries and not even approximate symmetries out of the asymptotic region.

In the following, the total space M (to be defined below) will simply be a product of real spaces, i.e. will have a trivial topology. It would obviously be of interest—both theoretically and in view of practical applications—to consider topologically non-trivial situations. Anyway, we will not attempt such an analysis here, and possibly postpone this task to a later time. This is motivated not only by the need to analyse the simpler case of trivial topology before attempting to consider global phenomena, but also by the fact that the present approach is primarily intended to study physical phenomena for which the trivial topology of the total space is the natural one, such as the formation of patterns in open systems [24, 25]; actually, in some of these cases the main difficulty (e.g. the continuous spectrum in the bifurcation analysis of the Swift–Hohemberg equation [24, 25]) would disappear by imposing a non-trivial topology (e.g. imposing periodic or Neumann boundary conditions on a finite space domain).

The paper is organized as follows: in section 2 we recall how equations and functions are naturally seen as geometrical objects, i.e. manifolds in appropriate spaces; in section 3 we consider vector fields in the total space M (i.e. the space of dependent and independent variables) of a differential equation and its prolongation to the jet space; in section 4 we see how these vector fields induce an action in the spaces of sections of some fiber bundles, which are the natural ones to set our discussion in view of the construction of section 2: this will induce an action of vector fields defined on M to the associated spaces of functions and equations; we are thus ready to consider in section 5 the symmetry properties of functions and equations in this new geometrical setting. In section 6 we define and consider the *asymptotic symmetries* using some simple concepts from dynamical systems theory, and in section 7 we discuss how these can be of use in the analysis of asymptotic properties of equations and their solutions, together with some general remarks. In the final section 8 we present a variety of examples to illustrate our setting and results that can be obtained in it, both for dynamical systems and for the evolution PDEs. More complex applications will be presented in a separate work.

2. Equations and functions as geometrical objects

In order to fix notation, we will begin by shortly recalling some features of the LT approach to DEs.

We will denote by x the independent variables (at this stage we do not distinguish between space and time variables) and by u the dependent ones; x and u will belong respectively to the spaces X and U, which here we assume to be embedded smooth submanifolds of $\mathbf{R}^q$ and $\mathbf{R}^p$. The *total space* will be

$$M = X \times U \tag{1}$$

and we will consider the *jet spaces* [1, 26–28] associated to M; these can be seen locally as

$$J^nM = X \times U^{(n)} \tag{2}$$

where $U^{(n)}$ is the space of dependent variables and their partial derivatives of order up to n. The global geometry of J^nM depends on the geometry of X and U as well as on the boundary conditions imposed on the DE.

A DE will be written as

$$\Delta :\equiv F(x, u^{(n)}) = 0 \tag{3}$$

and is identified with the *solution manifold* S that it specifies in J^nM:

$$\Delta \Leftrightarrow S_\Delta = \{(x, u^n) : F(x, u^n) = 0\} \subseteq J^nM. \tag{4}$$

As for solutions to Δ, any function $f : X \to U$ is also identified with a manifold in M, its *graph* γ_f:

$$\gamma_f = \{(x, u) : u = f(x)\} \subseteq M. \tag{5}$$

Clearly, once we have f (or γ_f) we also have its *prolongation* $f^{(n)}$ (or $\gamma_f^{(n)} \equiv \gamma_{f^{(n)}}$), which is obtained by simply considering partial derivatives of f up to order n:

$$\gamma_f^{(n)} = \{(x, u^{(n)}) : u^{(n)} = f^{(n)}(x)\} \subseteq J^nM. \tag{6}$$

A function $f : X \to U$ is then a solution to Δ if and only if

$$\gamma_f^{(n)} \subseteq S_\Delta. \tag{7}$$

3. Vector fields on M

Let us now consider vector fields (VFs) on M, i.e. elements η of $\mathcal{M} = \mathcal{D}\mathrm{iff}(M)$. These will be written as

$$\eta = \xi^i(x, u)\frac{\partial}{\partial x^i} + \varphi^\alpha(x, u)\frac{\partial}{\partial u^\alpha}. \tag{8}$$

Such a VF can be *prolonged* to a VF $\eta^{(n)}$ on J^nM, by the standard *prolongation formula* [1–3]; in multi-index notation [1], this is

$$\eta^{(n)} = \eta + \sum_{|J|\leqslant n} \Phi_J^\alpha \frac{\partial}{\partial u_J^\alpha} \tag{9}$$

$$\Phi_J^\alpha = D_J\left[\varphi^\alpha - \xi^i u_i^\alpha\right] + \xi^i u_{i,J}^\alpha \tag{10}$$

where D is the total derivative.

Such a VF generates a one-parameter group of diffeomorphisms of M (and consequently of J^nM); notice that smooth submanifolds of M will be transformed into smooth submanifolds of M, but not necessarily graphs into graphs (see e.g. the discussion in [1]); indeed, a graph is better seen as a *section* of the bundle (M, X, π), with total space M, base space X and projection $\pi(x, u) = x$. Only elements of $\mathcal{M}$ preserving the fibered structure will transform graphs into graphs. These elements form the algebra

$$\mathcal{M}_G = \mathcal{D}\mathrm{iff}(X) \oplus_\rightarrow \mathcal{D}\mathrm{iff}(U) \tag{11}$$

which we will consider from now on; here $\oplus_\rightarrow$ denotes semidirect sum, and the subscript G stands for 'gauge' (see [22]).

Elements of $\mathcal{M}_G$ are written, in the notation already used in (8), in the form

$$\eta = \xi^i(x)\frac{\partial}{\partial x^i} + \varphi^\alpha(x, u)\frac{\partial}{\partial u^\alpha}. \tag{12}$$

4. Induced actions of $\mathcal{M}$ on functions and equations

Clearly, the action of $\eta \in \mathcal{M}_G$ induces an action in the space Γ of smooth sections of (M, X, π); under this a section $\gamma = \gamma_f$ will be transformed into a new section $\gamma' = \gamma_{f'}$; in other words, η induces an action in the space of smooth functions $f : X \to U$.

This action can be determined as follows: we have that points (x, u) are acted upon as

$$(I + \varepsilon\eta)(x, u) = (\widetilde{x}, \widetilde{u}) = (x + \varepsilon\xi(x), u + \varepsilon\varphi(x, u))$$

and we want to find an $\widetilde{f}$ such that $\widetilde{u} = \widetilde{f}(\widetilde{x})$. By the above formula, at first order in ε we have

$$\begin{aligned}\widetilde{u} = f(x) + \varepsilon\varphi(x, u) &= f(\widetilde{x} - \varepsilon\xi(x)) + \varepsilon\varphi(\widetilde{x} - \varepsilon\xi(x), u) \\ &= f(\widetilde{x}) + \varepsilon\left[\varphi(\widetilde{x} + \mathrm{O}(\varepsilon), f(\widetilde{x}) + \mathrm{O}(\varepsilon)) - (Df)\cdot\xi\right] + \mathrm{O}(\varepsilon^2) \\ &= f(\widetilde{x}) + \varepsilon\left[\varphi(\widetilde{x}, f(\widetilde{x})) - (\xi\cdot Df)\right]\end{aligned}$$

so that the action of η in Γ can be described by the formal VF $\widetilde{\eta} : \Gamma \to T\Gamma$ which acts as

$$\begin{aligned}&\left(\mathrm{e}^{\varepsilon\widetilde{\eta}} f\right)^\alpha(x) = f^\alpha(x) + \varepsilon\delta f^\alpha(x) + \mathrm{O}(\varepsilon^2) \\ &(\widetilde{\eta} f)^\alpha \equiv \delta f^\alpha(x) = \varphi^\alpha(x, f(x)) - \xi^i \frac{\partial f^\alpha(x)}{\partial x^i}.\end{aligned} \tag{13}$$

Note that if we introduce the space $\Gamma_S \subset \Gamma$ of smooth sections being the graph of solutions to Δ (i.e. of $\gamma \in \Gamma$ such that $\gamma \subset S_\Delta$), then for $\eta \in \mathcal{G}_\Delta$ we have $\widetilde{\eta} : \Gamma_S \to T\Gamma_S$; this is indeed a direct translation of the statement that a symmetry of the equation transforms solutions into solutions.

It should also be remarked that Γ is a smooth infinite dimensional manifold, in general not complete; this means that when considering the flow of diffeomorphisms on it we must pay attention to the meaningfulness of limits.†

In a similar way, the action of $\eta^{(n)}$ in $J^n M$ also transforms smooth submanifolds of $J^n M$—as the solution manifold S_Δ—into smooth submanifolds of $J^n M$; since equations are identified with their solution manifolds, this also induces an action in the set E^n_M of equations of order n with variables in M.

Here we are mainly interested in *evolution equations*, in which case some extra care must be taken, as we will now discuss.

We denote by x the spatial independent variables, and by t the time variable. An evolution equation will be one of the form

$$\Delta :\equiv u_t = G\left(x, t; u^{[n]}\right) \tag{14}$$

where σ_Δ is a smooth function of its arguments and $u^{[n]}$ represents dependent variables and their partial derivatives *with respect to space variables only* up to order n.

In this setting, $J^n M$ can be seen as a bundle whose fiber corresponds to the space of t-derivatives of the u's, and the base is the space of independent and dependent variables together with their derivatives of order up to n with respect to the space variables; this will be called the nth *space jet*, $J^n_x M$. With κ the projection $\kappa : J^n M \to J^n_x M$, the evolution equations in E^n_M are then in correspondence with elements of the space Σ of sections of $(J^n M, J^n_x M, \kappa)$. In other words, the evolution equation (14) is identified with the section $\sigma_\Delta \in \Sigma$ given by the graph of G.

† This non-completeness is due to the requirement of smoothness for section to be in Γ; in physical terms, if for $\eta \in \mathcal{G}_\Delta$, the flow of $\widetilde{\eta}$ in Γ drives γ_f to $\partial\Gamma/\Gamma$; this means that f is transformed into a singular solution.

Remark. Note that in this setting the section σ_Δ corresponds to the solution manifold S_Δ, i.e., a point $(\alpha, \beta) \in J^n M$, where $\alpha \in J^n_x M$, $\beta \in \kappa^{-1}(\alpha)$, is in S_Δ if and only if $\beta = \sigma_\Delta(\alpha)$.

Note also that now the condition $\eta \in \mathcal{M}_G$ is not sufficient to ensure that $\eta^{(n)} : \Sigma \to T\Sigma$; to achieve this, we must ask instead that $\eta \in \mathcal{M}^t_G \subset \mathcal{M}_G$, where $\mathcal{M}^t_G$ is given by

$$\mathcal{M}^t_G = [\mathcal{D}\mathrm{iff}(T) \oplus_\to \mathcal{D}\mathrm{iff}(X)] \oplus_\to \mathcal{D}\mathrm{iff}(U). \tag{15}$$

Here and in the following, $t \in T$, $x \in X$, and

$$M = (T \times X) \times U. \tag{1'}$$

Elements of $\mathcal{M}^t_G$ will be written in the form

$$\eta = \tau(t)\frac{\partial}{\partial t} + \xi^i(x,t)\frac{\partial}{\partial x^i} + \varphi^\alpha(x,t;u)\frac{\partial}{\partial u^\alpha}. \tag{16}$$

Now the action induced by η in Σ can be determined pretty much in the same way as done for $\widetilde{\eta}$ above. We write equations in the form (14) and obtain that

$$\begin{aligned} \left(\mathrm{e}^{\varepsilon\widehat{\eta}} G\right)^\alpha &= G^\alpha + \varepsilon(\delta G)^\alpha + O(\varepsilon^2) \\ (\delta G)^\alpha &= (\tau_t G^\alpha - \varphi^\alpha_t) + \tau\frac{\partial G^\alpha}{\partial t} + \xi^i\frac{\partial G^\alpha}{\partial x^i} + \Phi^\beta_J \frac{\partial G^\alpha}{\partial u^\beta_J} \end{aligned} \tag{17}$$

where we have used again the multi-index notation and Φ^β_J are given by the prolongation formula (10).

Summarizing the discussion conducted up to now, we have considered VFs in $\mathcal{M}$, and then restricted our attention to $\mathcal{M}^t_G \subset \mathcal{M}$; for any VF $\eta \in \mathcal{M}^t_G$, we have considered associated VFs $\eta^{(n)}$, $\widetilde{\eta}$, $\widehat{\eta}$, acting in the spaces as follows:

$$\begin{array}{cccccc} \eta & : & M & \to & TM \\ \eta^{(n)} & : & J^n M & \to & T(J^n M) \\ \widetilde{\eta} & : & \Gamma & \to & T\Gamma \\ \widehat{\eta} & : & \Sigma & \to & T\Sigma \end{array}$$

Then $\eta \in \mathcal{G}_\Delta \Leftrightarrow \widetilde{\eta} : \Gamma_S \to T\Gamma_S$.

5. Symmetries, and fixed points of the induced flows

In LT, it is customary to focus attention on the symmetries of equations and of solutions; according to the standard definitions [1–4], the symmetry algebra $\mathcal{G}_\Delta$ of the equation $\Delta \in E^n_M$ is the algebra

$$\mathcal{G}_\Delta = \{\eta \in \mathcal{M} \;/\; \eta^{(n)} : S_\Delta \to TS_\Delta\} \subseteq \mathcal{M}. \tag{18}$$

Similarly, for a function $f : X \to U$, the symmetry algebra $\mathcal{G}_f$ is defined as

$$\mathcal{G}_f = \{\eta \in \mathcal{M} \;/\; \eta : \gamma_f \to T\gamma_f\}. \tag{19}$$

As discussed above, in this context we actually want $\eta \in \mathcal{M}_G \subset \mathcal{M}$ in (18) and (19).

Note that for f a solution of Δ—i.e. such that $\gamma_f^{(n)} \subset S_\Delta$—it is not necessary that $\mathcal{G}_f \subseteq \mathcal{G}_\Delta$; the VFs which are in $\mathcal{G}_f$ for some solution f but not in $\mathcal{G}_\Delta$ correspond to *conditional symmetries* [10–18]. Physically, conditional symmetries correspond to the possibility of solutions having symmetries which are *not* a symmetry of the equation; for example, the equation

$$\left(\partial_x^2 + \partial_y^2\right) f = -2\alpha f + \alpha^2(x^2 + y^2) f + \beta y^2 f + (\beta/\alpha) y \partial_y f$$

(α, β are real constants) which is not symmetric under rotations in the (x, y) plane, admits the solution

$$f(x, y) = \mathrm{e}^{-\alpha(x^2+y^2)/2}$$

which is rotationally symmetric.

Now the symmetry conditions used in (18) and (19) to define $\mathcal{G}_\Delta$ and $\mathcal{G}_f$ can be expressed in terms of the VFs $\widetilde{\eta}$ and $\widehat{\eta}$ introduced above. Indeed, it is clear that

$$\mathcal{G}_\Delta = \{\eta / \mathrm{e}^{\lambda\widehat{\eta}} \sigma_\Delta = \sigma_\Delta \, \forall \lambda \in \boldsymbol{R}\} \tag{18$'$}$$

$$\mathcal{G}_f = \{\eta / \mathrm{e}^{\lambda\widetilde{\eta}} \gamma_f = \gamma_f \, \forall \lambda \in \boldsymbol{R}\} \tag{19$'$}$$

are equivalent to our previous definitions (18) and (19).

Let us look at this again but from a slightly different point of view. The VFs $\widetilde{\eta}$ and $\widehat{\eta}$ define a *dynamical system* [29–33] in the spaces Γ and Σ, which we write formally as

$$\dot{\gamma} = \widetilde{\eta}\gamma \tag{20}$$

$$\dot{\sigma} = \widehat{\eta}\sigma \tag{21}$$

(here the dot denotes differentiation with respect to an evolution parameter λ (see (18$'$), (19$'$)) and not with respect to the physical time); the evolution of (20) and (21) gives the solutions

$$\Phi(\lambda, \gamma_0) = \mathrm{e}^{\lambda\widetilde{\eta}} \gamma_0 \tag{22}$$

$$\Psi(\lambda, \sigma_0) = \mathrm{e}^{\lambda\widehat{\eta}} \sigma_0 \tag{23}$$

where we have used a notation which is standard in the dynamical systems theory: i.e. if $\gamma(\lambda_0) = \gamma_0$, then $\gamma(\lambda) = \Phi(\lambda - \lambda_0, \gamma_0)$, and likewise for $\sigma(\lambda)$. We can then restate our symmetry criterion as follows:

Symmetry criterion. A function f is symmetric under η if and only if the corresponding section $\gamma_f \in \Gamma$ is a fixed point for the flow generated by $\widetilde{\eta}$. An equation Δ is symmetric under η if and only if the corresponding section $\sigma_\Delta \in \Sigma$ is a fixed point for the flow generated by $\widehat{\eta}$.

The asymptotic symmetries will be defined by means of this formulation; to this aim we must consider the flows (20) and (21) in greater generality, i.e. not limiting our attention to the fixed points. The general theory of dynamical systems and flows is neatly exposed, for instance in [29–33], to which the reader is referred.

6. Asymptotic symmetries

As for every flow in finite or—as is the case here—infinite-dimensional case, we can associate at least a local stable, unstable or centre manifolds to every fixed point p_0 of the flows (20) and (21). These will be denoted $W_s(p_0)$, $W_u(p_0)$, $W_c(p_0)$, respectively, and are tangent in p_0 to the stable, unstable and centre eigenspaces, $L_s(p_0)$, $L_u(p_0)$, $L_c(p_0)$, respectively. These are the spaces spanned by the eigenvectors of $\mathcal{L}(p_0)$ with negative, positive, and zero real part eigenvalues, respectively; $\mathcal{L}(p_0)$ is the linearization of the relevant VF v at the point p_0 (for further details, including the difficulties that can be met in defining the above mentioned objects, we refer to the books quoted above [29–33]).

We recall that $W_s(p_0)$ and $W_u(p_0)$, respectively, are defined as the manifolds of points p such that the limit for $t \to \infty$ of the flow issuing from p and obeying $\dot{p} = v(p)$ is p_0; and respectively the limit for $t \to \infty$ of the time-reversed flow issuing from p and obeying $\dot{p} = -v(p)$ is p_0. One also says [29] that p_0 is the *ω-limit* for all points $p \in W_s(p_0)$, and the *α-limit* for all points $p \in W_u(p_0)$.

To rephrase it: for any $p \in W_s(p_0)$ and $\delta > 0$, there is $t > 0$ such that $|\varphi(t, p) - p_0| < \delta$, where φ is the flow starting from p at $t = 0$ and following $\dot{p} = v(p)$; and for $\forall p \in W_u(p_0)$ and $\delta > 0$, there is $t > 0$ such that $|\varphi^*(t, p) - p_0| < \delta$, where φ^* is the flow starting from p at $t = 0$ and following $\dot{p} = -v(p)$. (In the following, an upper $*$ denotes the time-reversed flow). This suggests and motivates the following:

Definition. The VF $\eta \in \mathcal{M}_G$ *is an* ω-asymptotic symmetry *of the function* $f : X \to U$ *if and only if* $\lim_{\lambda\to\infty} \Phi(\lambda, \gamma_f) = \gamma_f^{\infty}$ *is a fixed point under* $\widetilde{\eta}$, *i.e.* $e^{\lambda\widetilde{\eta}}\gamma_f^{\infty} = \gamma_f^{\infty}$ $\forall\lambda$. *The* VF $\eta \in \mathcal{M}_G$ *is an* α-asymptotic symmetry *of the function* $f : X \to U$ *if and only if* $\lim_{\lambda\to-\infty} \Phi(\lambda, \gamma_f) \equiv \lim_{\lambda\to\infty} \Phi^*(\lambda, \gamma_f) = \gamma_f^{-\infty}$ *is a fixed point under* $\widetilde{\eta}$, *i.e.* $e^{\lambda\widetilde{\eta}}\gamma_f^{-\infty} = \gamma_f^{-\infty}$ $\forall\lambda$.

Note that, since η is a diffeomorphism, it is legitimate to consider time-reversed flows, and γ_0 is a fixed point under $\widetilde{\eta}$ if and only if it is a fixed point under $-\widetilde{\eta}$. Note also that fixed points under $\widetilde{\eta}$ are the functions f such that $\widetilde{\eta}(\gamma_f) = 0$.

We also give a similar definition for the asymptotic symmetries of equations:

Definition. The VF $\eta \in \mathcal{M}$ *is an* ω-asymptotic symmetry *of the equation* $\Delta \in E_M^n$ *if and only if* $\lim_{\lambda\to\infty} \Psi(\lambda, \sigma_\Delta) = \sigma_\Delta^{\infty}$ *is a fixed point under* $\widehat{\eta}$, *i.e.* $e^{\lambda\widehat{\eta}}\sigma_\Delta^{\infty} = \sigma_\Delta^{\infty}$ $\forall\lambda$. *The* VF $\eta \in \mathcal{M}$ *is an* α-asymptotic symmetry *of the equation* $\Delta \in E_M^n$ *if and only if* $\lim_{\lambda\to-\infty} \Psi(\lambda, \sigma_\Delta) \equiv \lim_{\lambda\to\infty} \Psi^*(\lambda, \sigma_\Delta) = \sigma_\Delta^{-\infty}$ *is a fixed point under* $\widehat{\eta}$, *i.e.* $e^{\lambda\widehat{\eta}}\sigma_\Delta^{-\infty} = \sigma_\Delta^{-\infty}$ $\forall\lambda$.

The reason for the name 'asymptotic symmetry' is quite clear: e.g. if η is an ω-asymptotic symmetry of f, then under the flow of $\widetilde{\eta}$ the function f is driven to a function f_0 for which η is an ordinary symmetry, and likewise for σ_Δ.

Important remark. We stress that 'asymptotic' refers to the flows induced by the VF η. Asymptotic properties in this sense can correspond or not to asymptotic properties in time and/or space, depending on our choice of η.

7. Discussion

Let us now shortly discuss the utility and the practical use of the setting developed in sections 4–6. As we have seen, given a DE Δ_0, we are free to choose to consider a VF η which may be or may not be a symmetry VF for Δ_0; let us consider the two cases separately.

Case A.
Let us assume that η is a symmetry of Δ_0; that is,

$$\widehat{\eta}\Delta_0 = 0. \tag{24}$$

In this case the solution manifold $S_\Delta \subset J^n M$ is invariant under the flow of $\eta^{(n)}$; the space Γ_S of sections being the graph of solutions to Δ_0 (see section 4) is therefore also invariant under $\widetilde{\eta}$. Namely, (24) implies that

$$\eta^{(n)} : S_\Delta \to TS_\Delta; \widetilde{\eta} : \Gamma_S \to T\Gamma_S. \tag{25}$$

Let us denote all independent, i.e. spacetime, variables by x and assume that $\lim[e^{\lambda\eta}x; \lambda \to \pm\infty]$ describes the asymptotic region we are interested in, for all $x \in X$. Then the asymptotic behaviour of a solution $u = f(x)$ to Δ_0 is described by the limit $f_{\pm\infty}$ of $e^{\lambda\widetilde{\eta}} f$ for $\lambda \to \pm\infty$.

In particular, if the flow of $\widetilde{\eta}$ in $\Gamma_S \subset \Gamma$ happens to be ω-asymptotic or α-asymptotic, respectively, to a lower dimensional set $\Gamma_S^{(\omega)}$ or $\Gamma_S^{(\alpha)}$, respectively, the elements of this set give a classification of the asymptotic behaviour of solutions to Δ_0; in the renormalization group language, we say that they identify *universality classes*.

We have therefore the same setting (including motivation and use) as in the BKL analysis [19–21] of discrete RG approach to parabolic PDEs and asymptotic scaling behaviour of solutions. The present setting extends their idea to infinitesimal RG, and to general prescribed asymptotic behaviour.

Case B.
Let us now consider, for a DE Δ_0 with solution manifold $S_{\Delta_0} \equiv S_0$, a VF η which is *not* a symmetry of Δ_0; i.e. we now assume

$$\widehat{\eta}\Delta_0 \neq 0. \tag{26}$$

In this case, $\widehat{\eta}$ induces a non-trivial flow in the space of equations (see section 4); we can write

$$e^{\lambda\widehat{\eta}}\Delta_0 = \Delta_\lambda \tag{27}$$

and denote by $S_{\Delta_\lambda} \equiv S_\lambda$ the solution manifold of Δ_λ. As remarked in section 4, Δ_λ, S_λ and the associated section $\sigma_{\Delta_\lambda} \equiv \sigma_\lambda$ are realizations of the same geometrical object in different languages, and it is therefore clear that (27) implies

$$e^{\lambda\eta^{(n)}} S_0 = S_\lambda. \tag{28}$$

Let us now consider a solution f_0 to Δ_0, and denote by $\gamma_{f_0} \equiv \gamma_0$ its graph. We do now consider the flow originating from f_0 (that is, γ_0) under $\widetilde{\eta}$, and write

$$e^{\lambda\widetilde{\eta}} f_0 = f_\lambda. \tag{29}$$

We now denote by $\gamma_{f_\lambda} \equiv \gamma_\lambda$ the graph of f_λ. By virtue of (28),

$$\gamma_0 \subset S_0 \Longrightarrow \gamma_\lambda \subset S_\lambda \tag{30}$$

so that while transforming the equation, η transforms accordingly the solutions. By letting λ go to $\pm\infty$ in (27–29), and writing $\Delta_{\pm\infty}$, $f_{\pm\infty}$ for the limits if they exist (which we assume in the following), we have:

Lemma. *If f_0 is a solution to Δ_0 and the limits $f_{\pm\infty}$, $\Delta_{\pm\infty}$ exist, then $f_{\pm\infty}$ is a solution to $\Delta_{\pm\infty}$.*

Corollary. *If $\Delta_0 \to \Delta_*$ for $\lambda \to \pm\infty$ under (27), then all the solutions f_0 of Δ_0 go to solutions f_* of Δ_* under (29).*

This again permits us to classify equations in universality classes (identified by the limit equation Δ_*), and inside each class of equation we can give a classification of solutions in universality classes according to their limit behaviour.

Note, anyway, that if f_0 is a generic function such that f_* is a solution to Δ_* (with the above notation for these), f_0 is not necessarily a solution to Δ_0; namely, the fact that f_* is a solution to Δ_* is a necessary but not sufficient condition for f_0 to solve Δ_0.

In other words, if—given Δ_* and a solution f_*—we are able to determine the stable manifold W_* of f_* under the flow of $\widetilde{\eta}$, this will not automatically lead to solutions of Δ_0; on the other side, as stated in the corollary above, all solutions to Δ_0 (and any Δ_λ, for that matter) with prescribed asymptotic behaviour must lie in W_*, so that this can provide a useful reduction.

7.1. Some general remarks

It should be stressed that the classifications considered above are only able to capture those equations and functions whose asymptotic behaviour under the flow induced by the considered VF η is particularly simple—indeed, a fixed point. This behaviour is by no means a trivial feature, and indeed for a generic η we will have no solutions admitting such a limit; this should be no surprise, as it just means that no solutions have the asymptotic symmetry described by η, and we just have to try another VF.

While in same cases the kind of asymptotic behaviour one is interested in is given *a priori* (e.g. by physics modelled by the equations), it is a natural question to ask how one could classify the possible asymptotic behaviours of solutions to a given DE Δ_0 (maybe inside a certain class, see examples 3 and 4 below).

In the case A above, we can *a priori* give an answer, but this is of little practical use in general; we should: (i) consider—i.e. determine—the algebra $\mathcal{G}_{\Delta_0}$ of the Lie-point symmetry VF of Δ_0; (ii) select in $\mathcal{G}_{\Delta_0}$ such VF η that the flow of $\widetilde{\eta}$ in Γ admits fixed points; (iii) check if these fixed points correspond actually to solutions of Δ_0, i.e. if they lie on $\Gamma_S \subset \Gamma$. Clearly this procedure 'by exhaustion' is not easy to implement, unless either the algebra $\mathcal{G}_{\Delta_0}$ is small or we can restrict the case to some subalgebra by physics arguments (we note in passing that this is the case e.g. for the heat equation [1, 2]). The situation is even worse for the case B above, i.e. if we do not restrict to considering symmetries of the equation Δ_0.

While I am unable at the moment to propose any systematic approach to the determination of VFs which would be asymptotic symmetries of solutions to a given DE Δ_0, it is quite conceivable that such a systematic approach can be found along the lines of the well known methods for determining the possible exact symmetries of solutions to Δ_0, such symmetries being, or not being, symmetries of the differential equation itself; see e.g. [10–18].

It should also be stressed that our discussion has been purely formal; i.e. we have not discussed the existence of limits, convergence of series, smoothness of functions, etc; at each step, we have assumed the objects to exist and to behave in the proper way. The examples to be considered in section 8 will show that the frame developed here is not empty, and actually applies to relevant physical equations and types of asymptotic behaviour.

On the whole, what is proposed here is a framework to discuss general asymptotic properties of (solutions to) differential equations; it is hoped that this also permits one to connect the powerful results and techniques of symmetry analysis of differential equations [1–4, 11–18] on one side, and pattern formation theory [19–21, 24, 25] on the other.

8. Examples

At this point, we would like to consider a few examples in some detail.

Example 1. Let $X = U = \boldsymbol{R}$ and consider the equation $\dot{u} = -cu$, with c a positive real constant. The solutions are $f(t) = \mathrm{e}^{-ct}\alpha$, for $\alpha \in U = \boldsymbol{R}$. Let us consider the VF $\eta = -\partial_t$: by (13), $(\delta f)(t) = \partial f/\partial t$. Note that η corresponds to a symmetry of the equation, so the space of solutions is invariant under $\widetilde{\eta}$. If we index the solutions by α, i.e. $f_\alpha(t) = \mathrm{e}^{-ct}\alpha$, the flow induced by $\widetilde{\eta}$ in $\Gamma_S \subset \Gamma$ is simply

$$\frac{\mathrm{d}\alpha}{\mathrm{d}\lambda} = -c\alpha. \tag{31}$$

This shows that $\gamma_0 = 0$ is the only fixed point, and that all solutions are attracted to it. Indeed, $f_0(t) \equiv 0$ is the only solution which possesses time-translation invariance, and all solutions satisfy $\lim_{t\to\infty} f(t) = 0 =$.

Example 2. Let $X = \boldsymbol{R}$, $U \subseteq \boldsymbol{R}^p$; consider the p-dimensional autonomous dynamical system

$$\dot{u}^i = F^i(u). \tag{32}$$

A solution $u^i = f^i(t)$ satisfies $\mathrm{d}f^i(t)/\mathrm{d}t = F^i\,(f(t))$. Let us consider again the VF $\eta = -\partial_t$, which is again a symmetry of the equation; by (13), this acts on f as $(\delta f)(t) = \partial f/\partial t$, which means that the flow under $\widetilde{\eta}$ is

$$\frac{\mathrm{d}f^i}{\mathrm{d}\lambda} = F^i(f). \tag{33}$$

Again, ω-asymptotic symmetry under $\widetilde{\eta}$ and t-asymptotic invariance under ∂_t are equivalent, and only solutions $u^i = f^i(t)$ such that $\lim_{t\to\infty} f(t)$ is a zero of F have ∂_t as ω-asymptotic symmetry; only solutions $u = f(t)$ such that $\lim_{t\to-\infty} f(t)$ is a zero of F have ∂_t as α-asymptotic symmetry. Note that solutions $f(t)$ having η as both α- and ω-asymptotic symmetries are either stationary ones, either homoclinic or heteroclinic orbits connecting fixed points.

Example 3. Let us consider the autonomous, rotationally invariant, dynamical system

$$\begin{aligned} \dot{x} &= + x - \Omega y - (x^2 + y^2)x \\ \dot{y} &= + y + \Omega x - (x^2 + y^2)y \end{aligned} \tag{34}$$

and let us consider the family of VFs

$$\eta = \alpha\partial_t + \beta(x\partial_y - y\partial_x) \tag{35}$$

which are symmetries of the equation.

Here $X = \boldsymbol{R}$, $U = \boldsymbol{R}^2$, and Γ corresponds to the space of smooth functions $f : \boldsymbol{R} \to \boldsymbol{R}^2$; a solution f will also be denoted by $f = (x_f(t), y_f(t))$.

According to (13), the action of $e^{\lambda\widetilde{\eta}}$ in Γ is described by

$$\begin{aligned} \frac{dx_f}{d\lambda} &= -\alpha[1-(x^2+y^2)]x_f - (\beta - \alpha\Omega)y_f \\ \frac{dy_f}{d\lambda} &= -\alpha[1-(x^2+y^2)]y_f + (\beta - \alpha\Omega)x_f \end{aligned} \tag{36}$$

so that we have no non-trivial fixed point under $\widetilde{\eta}$, unless we choose $\beta = \alpha\Omega$. Doing this, the above equations reduce to

$$\frac{d}{d\lambda}\begin{pmatrix} x_f \\ y_f \end{pmatrix} = -\alpha[1-(x^2+y^2)]\begin{pmatrix} x_f \\ y_f \end{pmatrix} \tag{37}$$

and therefore all solutions admit $\eta_0 = \partial_t + \Omega(x\partial_y - y\partial_x)$ as an α-asymptotic symmetry, and $\eta_0^* = -\partial_t - \Omega(x\partial_y - y\partial_x)$ as an ω-asymptotic symmetry.

Note that indeed all non-trivial solutions of our dynamical system are attracted to the unit circle and asymptotically are rotations on this circle with angular velocity Ω.

Example 4. Let us consider, mostly in general, the dynamical system

$$\begin{aligned} \dot{x} &= f(r)x - \Omega(r)y \\ \dot{y} &= f(r)y + \Omega(r)x \end{aligned} \tag{38}$$

where $r = \sqrt{x^2+y^2}$, and $f(r)$, $\Omega(r)$ are smooth functions. The family of VFs

$$\eta = -\partial_t - \beta(x\partial_y - y\partial_x) \tag{39}$$

is indexed by the real constant β. Clearly, these are symmetries of our DS for any choice of $\beta, f(r), \Omega(r)$.

Proceeding as in the previous example, we get

$$\begin{aligned} \frac{dx_f}{d\lambda} &= +\beta y_f + \dot{x}_f = f(r)x_f + (\beta - \Omega(r))y_f \\ \frac{dy_f}{d\lambda} &= -\beta x_f + \dot{y}_f = f(r)y_f - (\beta - \Omega(r))x_f. \end{aligned} \tag{40}$$

These admit non-trivial fixed points in correspondence with solutions of

$$\begin{cases} f(r) = 0 \\ \Omega(r) = \beta \end{cases} \tag{41}$$

Note that indeed solutions to our DS are attracted to circles of radius r_0 such that $r_0 f(r_0) = 0$, and on these we just have uniform rotation with angular velocity $\Omega(r_0)$.

Note also that in η_β we could equally well promote β to be an arbitrary smooth function of r (this corresponds to a module structure of the symmetry algebra, see [34]); in this case the final condition for η_β to be an asymptotic symmetry would still be the existence of r_0 such that

$$f(r_0) = 0 \qquad \Omega(r_0) = \beta(r_0). \tag{42}$$

The flow of $\widetilde{\eta}$ will be α- or ω-asymptotic to the corresponding solutions

$$x_{\pm\infty} = r_0\cos[\Omega(r_0)t + \delta] \qquad y_{\pm\infty} = r_0\sin[\Omega(r_0)t + \delta] \tag{43}$$

according to the sign of $f'(r_0)$: indeed, we have $\dot{r} = rf(r)$.

Example 5. Let us finally consider evolution PDEs; we distinguish time and space variables and take $t \in T = \boldsymbol{R}_+$, $x \in X = \boldsymbol{R}$, $u \in U = \boldsymbol{R}$. Let us consider the VF

$$\eta = -2t\partial_t - x\partial_x + u\partial_u \tag{44}$$

which generates the scaling

$$t \to \widetilde{t} = \lambda^{-2}t \qquad x \to \widetilde{x} = \lambda^{-1}x \qquad u \to \widetilde{u} = \lambda u. \tag{45}$$

The effect of applying this to a function $f : T \times X \to U$ is readily evaluated: indeed, $e^{\lambda\eta}$ acts as in (45), so that $u = f(x,t)$ is transformed into

$$\widetilde{u} = \lambda u = \lambda f(x,t) = \lambda f(\lambda\widetilde{x}, \lambda^2\widetilde{t}) \equiv \widetilde{f}_{(\lambda)}(\widetilde{x}, \widetilde{t}) \tag{46}$$

or, with the notation introduced in section 4,

$$e^{\lambda\widetilde{\eta}} f = \widetilde{f}_{(\lambda)} \qquad \widetilde{f}_{(\lambda)}(x,t) = \lambda f(\lambda x, \lambda^2 t). \tag{47}$$

It is clear that the limit $f_\infty = \lim(\widetilde{f}_{(\lambda)}, \lambda \to \infty)$ captures the asymptotic behaviour of $f(x,t)$ in time and space (i.e. for $t \to \infty$, $|x| \to \infty$).

Note that by considering the VF $\eta_* = -\eta$ we would have captured the asymptotic behaviour of $f(x,t)$ for small x and t.

The fixed points under $\widetilde{\eta}$ are functions which obey the scaling laws (45), as it is the case for the fundamental solution f_0 of the heat equation

$$f_0(x,t) = \frac{1}{\sqrt{Dt}} \exp\left[-x^2/(Dt)\right]. \tag{48}$$

We can now compute the action of $\widetilde{\eta}$ by (13) or by (47); this gives

$$\widetilde{\eta} f = (\delta f)(x,t) = f + 2t\partial_t f + x\partial_x f \tag{49}$$

(one can easily check that this indeed gives $\widetilde{\eta} f_0 = 0$).

Let us consider specifically autonomous evolution equations of second order in space variables, i.e. in E^2_M; we write them in the form

$$u_t = u_{xx} + F(u, u_x, u_{xx}) \tag{50}$$

and restrict to the case of F polynomial (say with $F(0,0,0) = 0$), i.e.

$$F(u, u_x, u_{xx}) = \sum c_{\alpha\beta\gamma} u^\alpha u_x^\beta u_{xx}^\gamma \equiv \sum c_{\alpha\beta\gamma} |\alpha, \beta, \gamma\rangle \tag{51}$$

where α, β, γ run over non-negative integers ($\alpha+\beta+\gamma > 0$), and in the RHS term we have introduced summation over repeated indices and an obvious notation for the monomials in u, u_x, u_{xx}.

Elements of E^2_M are now identified by F, or equivalently by the coefficients $c_{\alpha\beta\gamma}$ in the expansion of F in terms of the basis functions $|\alpha, \beta, \gamma\rangle$. By (17), the (linear) action of $\widehat{\eta}$ on F is given by

$$\widehat{\eta} F \equiv (\delta F) = -2F + u\frac{\partial F}{\partial u} + 2u_x\frac{\partial F}{\partial u_x} + 3u_{xx}\frac{\partial F}{\partial u_{xx}}. \tag{52}$$

Note that the heat equation—which corresponds to $F = 0$—is a fixed point for the action of $\widehat{\eta}$, as it should be.

Using (52), we also immediately read the action of $\widehat{\eta}$ on $|\alpha, \beta, \gamma\rangle$: this is given by

$$\widehat{\eta}|\alpha, \beta, \gamma\rangle = (-2 + \alpha + 2\beta + 3\gamma)|\alpha, \beta, \gamma\rangle. \tag{53}$$

In other words, $|\alpha, \beta, \gamma\rangle$ are a diagonal basis for the flow of $\widehat{\eta}$; at the fixed point $F = 0$, the stable eigenspace is spanned by $|1,0,0\rangle$, there is a centre eigenspace spanned by $|2,0,0\rangle$ and $|0,1,0\rangle$, and all other directions are unstable. (In considering $\eta_* = -\eta$, all the signs and therefore stabilities would be reversed).

Example 6. Let us consider again evolution equations in 1+1 dimensions, $t \in T = \mathbf{R}$, $x \in X = \mathbf{R}$, $u \in U = R$ (note that t now spans the entire real line), and equations of the form

$$u_t = \Delta u + F(u) \tag{54}$$

with $F(u)$ a polynomial and $F(0) = 0$. The Ginzburg–Landau equation is an example of this for $F(u) = u - |u|^2 u$.

Let us consider the vector field

$$\eta = -\partial_t - v\partial_x \tag{55}$$

which generates the linear shift

$$t \to \widetilde{t} = t - \lambda \qquad x \to \widetilde{x} = x - \lambda v \qquad u \to \widetilde{u} = u. \tag{56}$$

Proceeding as in example 5, we see that

$$\mathrm{e}^{\lambda\widetilde{\eta}} f = \widetilde{f}_{(\lambda)} \qquad \widetilde{f}_{(\lambda)}(x,t) = f(x - \lambda v, t - \lambda). \tag{57}$$

We could also obtain, directly from (13), the (partial) differential equation for the flow of $\widetilde{f}_{(\lambda)}$ under $\widetilde{\eta}$: writing $\Phi(x, t; \lambda) = \widetilde{f}_{(\lambda)}(x, t)$, we have

$$\frac{\partial \Phi}{\partial \lambda} = \frac{\partial \Phi}{\partial t} + v\frac{\partial \Phi}{\partial x}. \tag{58}$$

If, in particular, f was a solution to (54), we could use this to rewrite (58) as

$$\frac{\partial \Phi}{\partial \lambda} = \frac{\partial^2 \Phi}{\partial x^2} + v\frac{\partial \Phi}{\partial x} + F(\Phi) \equiv \Delta\Phi + (v \cdot \nabla)\Phi + F(\Phi). \tag{59}$$

Let us now consider fixed points under the action induced by η; it is clear that every autonomous equation—and therefore, in particular, any equation of the form (54)—is invariant under η, so that the action of $\widehat{\eta}$ on our class of equations will be trivial.

As for the action of $\widetilde{\eta}$, the fixed points are given by (58) asking $\partial\Phi/\partial\lambda = 0$; solving the associated characteristic equation, we see (certainly with no surprise) that invariant functions are of the form

$$f(x, t) = f(x - vt). \tag{60}$$

(These solve (54) if equation (59) with $\partial\Phi/\partial\lambda = 0$ is satisfied; alternatively, we can substitute (60) in (54) and get the same equation.)

If we now consider the full equation (58), we again have to solve the associated characteristic equation; we have then as a (functional) basis of conserved functions

$$\zeta_1(x, t, \lambda) \equiv \xi = x - vt \qquad \zeta_2(x, t, \lambda) \equiv \tau = t + \lambda \tag{61}$$

as indeed expected from (57); note that $\zeta_1 - v\zeta_2 = x - \lambda v$. Therefore, solutions to (58) can be rewritten as

$$\Phi(x, t, \lambda) = \Psi(x - vt, t + \lambda) \equiv \Psi(\xi, \tau) \tag{62}$$

and functions invariant under $\widetilde{\eta}$ give $\partial\Psi/\partial\tau = 0$, see (60). Indeed, if we single out the λ dependence of Φ, we find that, with L being the Lie derivative,

$$L_\eta \Psi \equiv \Psi_\tau . \tag{63}$$

Note that this means in particular that if we Fourier-transform Ψ in τ, with the obvious notation

$$\Psi(\xi, \tau) = \chi_\omega(\xi)|\omega\rangle \tag{64}$$

then the basis functions $|\omega\rangle = \mathrm{e}^{i\omega\tau}$ undergo a periodic evolution under the flow of η. Note also that if the expansion (64) has non-zero coefficients χ_ω for different ω's which are not rationally dependent, the evolution of Ψ is quasi-periodic.

The evolution of Φ under η is particularly simple if we can factorize Ψ as

$$\Psi(\xi, \tau) = A(\tau)B(\xi) \tag{65}$$

which corresponds to the functions $f(x, t)$, considered above, of the form

$$f(x, t) = a(x) f_0(x - vt). \tag{66}$$

With the ansatz (65), we have $L_\eta\Psi \equiv \Psi_\tau = A_\tau B$; if $\Psi(\xi, \tau) \neq 0$, and rewriting $A_\tau = k(\tau)A$, we get

$$\frac{\mathrm{d}\Psi}{\mathrm{d}\tau} = k(\tau)\Psi \tag{67}$$

so that e.g. any solution of the form $f(x, t) = \mathrm{e}^{-P(x^2)} f_0(x - vt)$ (where P is a positive polynomial) is driven by η to the null solution. More in general, we obtain that if Ψ can be written as a converging series

$$\Psi(\xi, \tau) = \Psi_0(\xi) + \sum_{m=1}^{\infty} A_m(\tau)\Psi_m(\xi) \tag{68}$$

with A_m satisfying $A_m(\tau)\cdot(\mathrm{d}A_m(\tau)/\mathrm{d}\tau) < 0$, then $\lim[\Phi(x, t, \lambda); \lambda \to \pm\infty] = \Phi_\pm(x - vt)$.

Correspondingly, if we write f in the form (cf [24], section 28)

$$f(x, t) = f_0(x - vt) + \sum_{\ell=1}^{\infty} a_\ell(x) f_\ell(x - vt) \tag{69}$$

we see that $\lim[f_{(\lambda)}(x, t); \lambda \to \infty] = f_0(x - vt)$ exactly, if and only if $(a_\ell(x)\cdot \mathrm{d}a_\ell(x)/\mathrm{d}x) < 0$. If we consider the time-reversed vector field $\eta_* = -\eta$, we should demand instead $(a_\ell(x) \cdot \mathrm{d}a_\ell(x)/\mathrm{d}x) > 0$.

References

[1] Olver P J 1986 *Applications of Lie Groups to Differential Equations* (Berlin: Springer)

[2] Bluman G W and Kumei S 1989 *Symmetries and Differential Equations* (Berlin: Springer)

[3] Stephani H 1989 *Differential Equations. Their Solution Using Symmetries* (Cambridge: Cambridge University Press)

[4] Ovsjannikov L V 1962 *Group Properties of Differential equations* (Novosibirsk); 1982 *Group Analysis of Differential Equations* (New York: Academic)

[5] Bluman G W and Cole J D 1974 *Similarity Methods for Differential Equations* (Berlin: Springer)

[6] Hill J M 1982 Solution of differential equations by means of one-parameter groups *Res. Notes Math.* **63** (London: Pitman)

[7] Olver P J 1979 appendix to Sattinger D H Group theoretic methods in bifurcation theory *Lect. Notes in Math.* **762** (Berlin: Springer)

[8] Hamermesh M 1984 The symmetry group of a differential equation *Group Theoretical Methods in Physics (Proc. XII ICGTMP, Trieste) Lecture Notes in Physics 201* ed G Denardo *et al* (Berlin: Springer)

[9] Winternitz P 1987 What is new in the study of differential equations by group theoretical methods *Group Theoretical Methods in Physics* (Proc. XV ICGTMP, Philadelphia) ed R Gilmore (Singapore: World Scientific)

[10] Olver P J and Rosenau P 1986 *Phys. Lett.* **114A** 107; 1987 *SIAM J. Appl. Math.* **47** 263

[11] Bluman G W and Cole J D 1969 *J. Math. Mech.* **18** 1025

[12] Clarkson P A and Kruskal M D 1989 *J. Math. Phys.* **30** 2201

[13] Levi D and Winternitz P 1989 *J. Phys. A: Math. Gen.* **22** 2915

[14] Pucci E and Saccomandi G 1992 *J. Math. Anal. Appl.* **163** 588

[15] Pucci E 1992 *J. Phys. A: Math. Gen.* **25** 2631

[16] Vorob'ev E M 1986 *Soviet Math. Dokl.* **33** 408

[17] Vorob'ev E M 1991 *Acta Appl. Math.* **23** 1

[18] Olver P J 1993 Direct reduction and differential constraints *preprint*

[19] Bricmont J and Kupiainen A 1992 *Comm. Math. Phys.* **150** 193

[20] Bricmont J, Kupiainen A and Lin G 1993 *Commun. Pure Appl. Math.* at press

[21] Bricmont J, Kupiainen A and Lin G 1992 Stability of moving fronts in the Landau–Ginzburg equation *Preprint*

[22] Gaeta G 1992 *Helv. Phys. Acta* **65** 622; 1992 *Acta Appl. Math.* **28** 43; 1993 *Lett. Math. Phys.* **28** 1

[23] Baikov V A, Gazizov R K and Ibragimov N K 1991 *J. Soviet Math.* **55** 1450

[24] Collet P and Eckmann J P 1990 *Instabilities and Fronts in Extended Systems* (Princeton: Princeton University Press)

[25] Collet P and Eckmann J P 1990 *Comm. Math. Phys.* **132** 139 1992 *Commun. Math. Phys.* **145** 345; 1992 *Nonlinearity* **5** 1265;
Collet P, Eckmann J P and Epstein H 1992 *Helv. Phys. Acta* **65** 56

[26] Arnold V I 1989 *Mathematical Methods of Classical Mechanics* (Berlin: Springer) 2nd edn; 1983 *Geometrical Methods in the Theory of Ordinary Differential Equations* (Berlin: Springer); 1990 *Contact Geometry and Wave Propagation* (Geneva: Les Editions de L'Enseignement Mathematique)

[27] Ehresmann C 1951 *C. R. Acad. Sci. Paris* **233** 598, 777, 1081; 1952 *C. R. Acad. Sci. Paris* **234** 1028, 1424, 587; 1954 *C. R. Acad. Sci. Paris* **239** 1762; 1955 *C. R. Acad. Sci. Paris* **240** 397, 1755

[28] Saunders D J The geometry of jet bundles 1990 *London Mathematical Society Lecture Notes Series* **142** (Cambridge: Cambridge University Press)

[29] Hirsch M and Smale S 1974 *Differential Equations, Dynamical Systems and Linear Algebra* (New York: Academic Press)

[30] Guckenheimer J and Holmes P 1983 *Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields* (Berlin: Springer)

[31] Ruelle D 1989 *Elements of Differential Dynamics and Bifurcation Theory* (London: Academic)

[32] Verhulst F 1989 *Nonlinear Differential Equations and Dynamical Systems* (Berlin: Springer)

[33] Arnold V I 1992 *Ordinary Differential Equations* 2nd edn (Berlin: Springer)

[34] Cicogna G and Gaeta G 1992 *Nuovo Cim.* B **107** 1085; 1992 *Ann. Inst. H. Poincaré* **56** 375
Gaeta G 1993 *Int. J. Theor. Phys.* **32** 191